M. Plana with the Author's Comp.

5

OBSERVATIONS

ON THE

NOTATION EMPLOYED

IN THE

CALCULUS OF FUNCTIONS.

From the Transactions *of the* Cambridge Philosophical Society.

By CHARLES BABBAGE, Esq. A.M.

FELLOW OF THE ROYAL SOCIETIES OF LONDON AND EDINBURGH,
AND OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

CAMBRIDGE:

Printed by J. Smith, Printer to the University.

1820

V. *Observations on the Notation employed in the Calculus of Functions.*

By CHARLES BABBAGE, M.A.

FELLOW OF THE ROYAL SOCIETIES OF LONDON AND EDINBURGH,

AND OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

[Read *May* 1, 1820.]

AMONGST the various causes which combine in enabling us by the use of analytical reasoning to connect through a long succession of intermediate steps the data of a question with its solution, no one exerts a more powerful influence than the brevity and compactness which is so peculiar to the language employed. The progress of improvement in leading us from the simpler up to the most complex relations has gradually produced new modes of shortening the ancient paths, and the symbols which have thus been invented in many instances from a partial view, or for very limited purposes, have themselves given rise to questions far beyond the expectations of their authors, and which have materially contributed to the progress of the science. Few indeed have been so fortunate as at once to perceive all the bearings and foresee all the consequences which result either necessarily or analogically even from some of the simplest improvements.

The first analyst who employed the very natural abbreviation of a^2 instead of aa little contemplated the existence of fractional negative and imaginary exponents, at the moment when he

adopted this apparently insignificant mode of abridging his labor. So great however is the connection that subsists between all branches of pure analysis, that we cannot employ a new symbol or make a new definition, without at once introducing a whole train of consequences, and in defiance of ourselves, the very sign we have created, and on which we have bestowed a meaning, itself almost prescribes the path our future investigations are to follow.

Such being the power and influence of those symbols* by which mathematical reasoning is carried on, it cannot be considered as unimportant either as regards the particular branch, or with reference to the science in general, to examine some of the bearings of the notation which has been employed in the calculus of functions, and to resolve some of the unusual questions which it presents.

That the results to which such an enquiry will conduct us are of a nature purely speculative, is an objection to which every attempt to improve notation is liable; it can however never be considered an useless task to examine and strengthen that which essentially contributes to the power of an instrument, which enables us so wonderfully to trace the connexion between the phenomena of nature.

When it became convenient to express without performing the repetition of an operation, whose characteristic is f, the method which first presented itself doubtless was similar to that which had been adopted with such advantage for the exponents of quantities, and $f^2(x)$ was written instead of $ff(x)$; it now followed without any other convention that $f^3(x)$ and $f^n(x)$ represented $fff.(x)$, and $fff.$ (n times) (x) and also that

$$f^{n+m}(x) = f^n f^m(x) \; \ldots\ldots (A),$$

when n and m are whole numbers.

* Euler, to whom analysis is so much indebted, appears to have been fully aware of the power and importance of notation; "Summa analyseos inventa maximam partem algorithmo ad certas quasdem quantitates accommodato innitantur." Specimen Algorithmi singularis. Acad. Petrop. Comm. *Nov.* 1762.

At this point of generalization a question occurred as to the meaning of f^n when n is a fractional, surd, or negative number, and in order to determine it, recourse was had to a new convention not inconsistent with, but comprehending in it the former one. The index n was now defined by means of the equation (A) and was said to indicate such a modification of the function to which it is attached that that equation shall be verified.

From this extended view of the equation (A) several curious results follow; if $n=0$, it becomes

$$f^m(x) = f^0 f^m(x).$$

This informs us that f^0 is such an operation that when performed on any quantity it does not change it, or putting $f^m(x)=y$, it gives

$$f^0(y) = y,$$

a result which is analogous to $x^0=1$.

Let $m=-1$, $n=1$, we have

$$f^0 x = f^1 f^{-1}(x), \quad \text{or } f(f^{-1}x) = x;$$

$f^{-1}(x)$ must therefore signify such a function of x, that if we perform upon it the operation denoted by f it shall be reduced to x. The number of functions possessing this property will depend on the nature of f; thus if $f(x)=x^n$, let $r_1, r_2, \ldots r_n$ be the roots of $v^n-1=0$; then $f^{-1}(x)$ may be either of the n quantities

$$r_1 x^{\frac{1}{n}}, \; r_2 x^{\frac{1}{n}}, \ldots r_n x^{\frac{1}{n}};$$

for if we perform the operation f upon any of them, or raise it to the nth power, the result is x.

Here then we find

$$f(f^{-1}x) = x,$$

in all cases; but if the negative index is attached to the first functional sign, we have

$$f^{-1}(fx) = r_i x,$$

and one only of these values gives $f^{-1}fx = x$.

It was more necessary to make this observation, because several errors have arisen from not attending to it, and because that particular form of f^{-1} which gives $f^{-1}fx=x$ possesses peculiar properties: $f^{-1}(x)$ is then the inverse function of fx; and if we have the equation $f(x)=y$, we may indicate its resolution thus

$$x=f^{-1}(y).$$

Having established the connexion between positive and negative indices in functions of one variable, I shall now proceed to those containing two.

If in the function $\psi(x,y,)$ we substitute at the same time $\psi(x,y,)$ for x and y it becomes

$$\psi\{\psi(x,y),\ \psi(x,y)\},$$

and for the sake of brevity this has been denoted thus

$$\psi^{\overline{2,2}}(x,y):$$

similarly, if in this we wrote $\psi(x,y)$ for x and y we should have $\psi^{\overline{3,3}}(x,y)$, and a few steps would lead us to the equation,

$$\psi^{\overline{n,n}}\{\psi^{\overline{m,m}}(x,y),\ \psi^{\overline{m,m}}(x,y)\}=\psi^{\overline{n+m,n+m}}(x,y)\ldots\ldots(B).$$

The same reasons which in the former instance induced us to generalize the first definition, now point out the propriety of assuming this as the definition of simultaneous functions, and of the modifications implied by their indices. The first step is to discover the value of $\psi^{\overline{0,0}}(x,y)$; for this purpose put $n=0$, then

$$\psi^{\overline{0,0}}\{\psi^{\overline{m,m}}(x,y),\ \psi^{\overline{m,m}}(x,y)\}=\psi^{\overline{m,m}}(x,y),$$

and substituting v instead of $\psi^{\overline{m,m}}(x,y)$, we have

$$\psi^{\overline{0,0}}(v,v)=v.$$

If we had put $m=0$, $n=1$ instead of $n=0$, we should have found

$$\psi^{\overline{1,1}}\{\psi^{\overline{0,0}}(x,y),\ \psi^{\overline{0,0}}(x,y)\}=\psi^{\overline{1,1}}(x,y),$$

which shews that $\psi^{\overline{0,0}}(x,y)$ is such a function of x and y that when simultaneously substituted in $\psi(x,y)$, for x and y, it shall give either x or y: it may have different values like all other inverse functions.

If in the function $\psi(x, y)$, we put $\psi(x, y)$ instead of x it becomes $\psi\{\psi(x, y), y\}$ which is written thus $\psi^{2,1}(x, y)$, and continuing such substitutions we arrive at

$$\psi^{n,1}\{\psi^{m,1}(x, y), y\} = \psi^{n+m,1}(x, y) \ldots\ldots\ldots (C),$$

or if the n^{th} function relative to x, and then the m^{th} function relative to y be substituted, it would be

$$\psi^{n,1}\{x, \psi^{1,m}(x, y)\} = \psi^{n,m+1}(x, y) \ldots\ldots\ldots (D).$$

These two equations are now considered as the definition by which the meaning of the indices is known, and we proceed to enquire their values in particular cases: let $n = 0$ in the first and we have

$$\psi^{0,1}\{\psi^{m,1}(x, y), y\} = \psi^{m,1}(x, y);$$

put

$$\psi^{m,1}(x, y) = v, \text{ and it becomes}$$

$$\psi^{0,1}(v, y) = v, \text{ or } \psi^{0,1}(x, y) = x \ldots\ldots\ldots (E).$$

If $n = 1$ and $m = 0$ in the latter,

$$\psi^{1,1}\{x, \psi^{1,0}(x, y)\} = \psi^{1,1}(x, y),$$

whence it follows that

$$\psi^{1,0}(x, y) = y \ldots\ldots\ldots\ldots\ldots\ldots (F).$$

If $n = -m = 1$, (D) becomes

$$\psi^{1,1}\{x, \psi^{1,-1}(x, y)\} = \psi^{1,0}(x, y) = y,$$

hence $\psi^{1,-1}(x, y)$ expresses such a function of x and y that when substituted in $\psi(x, y)$ for y it reduces it to y, or if

$$\psi(x, y) = v,$$

then

$$y = \psi^{1,-1}(x, v) \text{ and } x = \psi^{-1,1}(v, y) \ldots\ldots\ldots\ldots (G).$$

From (C) we may readily deduce as consequences,

$$\psi^{0,n}(x, y) = x, \text{ and } \psi^{n,0}(x, y) = y.$$

The same reasoning which has been employed in discovering the meaning of the indices of functions of two variables applies with

equal facility to those of many variables; without repeating it, it is sufficient to observe that

$$\psi^{0,1,1,\ldots}(x, y, \ldots)=x, \quad \psi^{1,0,\ldots}(x, y, \ldots) = y, \text{ \&c.}$$

and that of

$$\psi^{1,1,1,\ldots}(x, y, z, \ldots) = v;$$

then

$$y = \psi^{1,-1,1,}(x, v, z, \ldots).$$

Amongst the questions which arise from this notation there are two which are altogether different from any which have occurred in others, and although their solution is not attended with any very great difficulty they furnish an enquiry of some curiosity.

If the expression $\psi^{n,m}(x, y)$ be fully written out, we may seek, first, *how many times will either of the quantities* x *or* y *be repeated;* and secondly, *how many times will the symbol* ψ *occur.*

I shall begin with some of the more simple questions, and first

PROB. I.

How often does the quantity x occur in the expression

$$\psi^{n,n}(x, y)?$$

Suppose it occurs u_n times, then in

$$\psi^{\overline{n+1,n+1}}(x, y) = \psi^{\overline{n,n}}\{\psi(x, y), \psi(x, y)\},$$

it occurs twice as many times as it does in the preceding; hence

$$u_{n+1} = 2u_n, \text{ or } u_n = C.2^n;$$

if $n=1$, $u_1=2C=1$, hence $C=\frac{1}{2}$, and $u_n=2^{n-1}$, or x occurs 2^{n-1} times in $\psi^{\overline{n,n}}(x, y)$.

PROB. II.

How many times is the symbol ψ repeated in $\psi^{\overline{n,n}}(x, y)$?

Let u_n be the number of times; then it is found u_{n+1} times in $\psi^{\overline{n+1,n+1}}(x, y)$; but in going from the first of these to the second we augment the number of times ψ occur by 2^n, because wherever x

occurs, we introduce ψ by putting $\psi(x, y)$ for it, and as x occurs 2^{n-1} times we by this means add 2^{n-1} times ψ, and similarly, on account of y we add 2^{n-1} times y, so that

$$u_{n+1} = u_n + 2^{n-1} + 2^{n-1} = u_n + 2^n,$$

$$\Delta u_n = 2^n, \text{ whence } u_n = 2^n + C,$$

if $n=1$ $u_1 = 2 + C = 1$, or $C = -1$,

therefore ψ occurs $2^n - 1$ times in $\psi^{n,n}(x, y)$.

Prob. III.

How often is x_1 repeated in $\psi^{\overline{n,n,\ldots}}(x_1, \ldots x_i)$?

Suppose it to occur u_n times, then in going from the n^{th} to the $n+1^{\text{th}}$ simultaneous function wherever x_1, $x_2, \ldots x_i$ occur, we introduce x_1 once, but each of these occurred u_n times, and as there are i of them, we have

$$u_{n+1} = iu_n, \text{ or } u_n = Ci^n;$$

but if $n=1$, $u_1 = Ci = 1$, wherefore $C = \frac{1}{i}$, and x occurs i^{n-1} times.

Cor. As $x_i, \ldots x_k$ are all similarly involved, they each of them occur i^{n-1} times.

Prob. IV.

How often does ψ occur in $\psi^{\overline{n,n\ldots}}(x_1, x_2, \ldots x_i)$?

Let it occur u_n times; then in the $(n+1)^{\text{th}}$ besides the u_n times that it occurs in the n^{th} we introduce an additional ψ wherever x_1, $x_2, \ldots x_i$ occur, or since there are i of these quantities, and each occurs i^{n-1} times, we have

$$u_{n+1} = u_n + i^n, \text{ whence } u_n = \frac{i^n}{i-1} + C,$$

$$\text{if } n=1,\ u_1 = \frac{i}{i+1} + C = 1, \text{ hence } C = \frac{-1}{i-1};$$

$$\text{and } \psi \text{ occurs } \frac{i^n - 1}{i - 1} \text{ times in } \psi^{n,n,\ldots}(x_1, \ldots x_i).$$

Prob. V.

How many times do x and y occur in $\psi^{n,1}(x, y)$?

In passing from the n^{th} function to the $n+1^{\text{th}}$ we change x into $\psi(x, y)$; this does not increase the number of times x occurs, and as the same reasoning applies to the $\overline{n-1}^{\text{th}}$, $\overline{n-2}^{\text{th}}$, ... down to the first it follows that x only occurs once in $\psi^{n,1}(x, y)$.

Again, in passing from the n^{th} to the $n+1^{\text{th}}$ we introduce y once, if therefore u_n represent the number of times it occurs, we have

$$u_{n+1} = u_n + 1, \text{ or } u_n = n + C,$$

if $n = 1$, $u_1 = 1 + C = 1$, or $C = 0$, and y occurs n times in $\psi^{n,1}(x, y)$.

And similarly, x occurs n times, and y once in $\psi^{1,n}(x, y)$.

Prob. VI.

How frequently does ψ occur in $\psi^{n,1}(x, y)$?

$$\text{since } \psi^{n+1,1}(x, y) = \psi^{n,1}\{\psi(x, y), y\},$$

each step introduces ψ once, it will therefore be repeated n times.

Prob. VII.

In $\psi^{n,m}(x, y)$ how often are x and y repeated?

$$\psi^{n,m}(x, y) = \psi^{n,1}\{x, \psi^{1,m-1}(x, y)\}.$$

In the right side of this equation x occurs once, and in $\psi^{n,m-1}(x, y)$ it is found $(m-1)$ times; now in each of these last x occurs $m-1$ times by Prob. 5, so that on the whole it appears $1 + n.(m-1) = mn - n + 1$ times.

Again, y occurs once in $\psi^{1,m}(x, y)$ and this latter is found n times in $\psi^{n,m}(x, y)$, so that y occurs n times in the same expression.

Prob. VIII.

How many times does ψ occur in $\psi^{n,m}(x, y)$?

In $\psi^{n,1}(x, y)$, ψ is found n times by Prob. 6; and y is found n

times; if for y we put $\psi^{1, m-1}(x, y)$ which contains ψ, $m-1$ times repeated, we shall have

$$\psi^{n,1}\{x, \psi^{1,m-1}(x, y)\} = \psi^{n,m}(x, y),$$

and the number of times ψ occurs is $n+n(m-1)=nm$ times.

Prob. IX.

How often do $x_1, x_2, \ldots x_k$ occur in $\psi^{a,b,c\cdots}(x_1, x_2, \ldots x_k)$?

First, let us consider $\psi^{0,1,1,\cdots}(x_1, x_2, \ldots x_k)$, and let x_1 occur u_a times then since

$$\psi^{a+1,\,\cdots}(x_1, x_2, \ldots x_k) = \psi^{1,1,\cdots}\{\psi^{a,1,\cdots}(x_1, x_2, \ldots x_k), x_2, \ldots x_k\},$$

the number of times x_1 occurs in the first side of the equation is u_{a+1}; in the second side it occurs u_a times, therefore $u_{a+1}=u_a$, or $u_a=C$; but if $a=1$, $u_1=C=1$, therefore $u_a=1$, and x occurs only once. Again, let u_a now represent the number of times x_2 occurs, then in the second side x_2 occurs u_a times in $\psi^{a,1,1,\cdots}(x_1, x_2, \ldots x_k)$, and also once besides, so that

$$u_{a+1}=u_a+1, \text{ or } u_a=a+C,$$

if $a=1$, $C=0$, and the same result will be found for any other quantity x_i except x_1, hence in $\psi^{a,1,1,\cdots}(x_1, x_2, \ldots x_k)$,

$$\left.\begin{array}{lr} x_1 \quad \text{occurs} & 1 \\ x_2 \ldots\ldots\ldots\ldots & a \\ \ldots\ldots\ldots\ldots\ldots & \\ x_i \ldots\ldots\ldots\ldots & a \end{array}\right\} \text{times} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let us next consider $\psi^{a,b,\cdots}(x_1, x_2 \ldots x_k)$,

$$\psi^{a,b,1,\cdots}(x_1, x_2, \ldots x_k) = \psi^{a,1,1,}\{x_1, \psi^{1,b-1,1,\cdots}(x_1, x_2, \ldots x_k), x_3, \ldots x_k\}.$$

On the right side of this equation x_1 occurs in two places first by itself in which state by (1) it is only found once; and secondly, it occurs in the function $\psi^{1,b-1,\cdots}(x_1, x_2, \ldots x_k)$ where by (1) it is found

to recur $b-1$ times, but that function itself is repeated a times, therefore the whole number of times x_1 is repeated is

$$1 + a(b-1) = ab - a + 1:$$

x_2 only occurs under the last mentioned function, and there it is found only once, but as that function is repeated a times x_2 must occur a times: x_3 occurs in two places; in the first it is repeated $a(b-1)$ times, and in the second a times, so that on the whole x_3 is found

$$ab - a + a = ab \text{ times.}$$

x_i is found the same number of times.

Therefore in

$$\psi^{a,b,1,\dots}(x_1, x_2, \dots x_k),$$

$$\left.\begin{array}{ll} x_1 \text{ occurs} & ab-a+1 \\ x_2 \dots\dots\dots & a \\ \dots\dots\dots\dots & \\ x_i \dots\dots\dots & ab \end{array}\right\} \text{times}\dots\dots\dots\dots\dots\dots (2)$$

by considering the equation,

$$\psi^{a,b,c,1,\dots}(x_1, x_2, \dots x_k) = \psi^{a,b,1,\dots}\{x_1, x_2, \psi^{1,1,c-1,1,\dots}(x_1, x_2, \dots x_k), x_4, \dots x_k),$$

we shall find that when there are three indices a, b, c,

$$\left.\begin{array}{ll} x_1 \text{ occurs} & abc-a+1 \\ x_2 \dots\dots\dots & abc-ab+a \\ x_3 \dots\dots\dots & ab \\ \dots\dots\dots\dots & \\ x_i \dots\dots\dots & abc \end{array}\right\} \text{times}\dots\dots\dots\dots\dots\dots (3)$$

a similar process applied to $\psi^{a,b,c,d,1,\dots}(x_1, x_2, \dots x_k)$ would show that in it

$$\left.\begin{array}{ll} x_1 \text{ occurs} & abcd-a+1 \\ x_2 \dots\dots & abcd-ab+a \\ x_3 \dots\dots & abcd-abc+ab \\ x_4 \dots\dots & abc \\ \dots\dots\dots & \\ x_i \dots\dots & abcd \end{array}\right\} \text{times}\dots\dots\dots\dots (4),$$

and in $\psi^{a,\,b,\,c,\,d,\,e,\,1,\cdots}(x_1,\,x_2,\,\ldots\,x_k)$,

$$\left.\begin{array}{ll} x_1 \text{ occurs} & abcde-a+1 \\ x_2 \;\ldots\ldots & abcde-ab+a \\ x_3 \;\ldots\ldots & abcde-abc+ab \\ x_4 \;\ldots\ldots & abcde-abcd+abc \\ x_5 \;\ldots\ldots & abcd \\ x_i \;\ldots\ldots & abcde \end{array}\right\} \text{times} \ldots\ldots\ldots\ (5)$$

the law which these expressions follow is easily discovered from those which have been given, and by their assistance it may be shewn that in the expression

$$\psi^{a,\,b,\,c,\cdots}(x_1,\,x_2,\,x_3,\,\ldots\,x_k),$$

ψ will be repeated $abcd\ldots$ times.

If in a function of two variables we wish after taking the $a, b, c, \cdots^{\text{th}}$ functions relative to the variables $x_1,\,x_2,\ldots$ in order, again to take the p^{th} relative to the first, the q^{th} relative to the second, &c. some modification of our notation becomes necessary. I have proposed in the Philosophical Transactions to make use of two rows of indices the upper to signify the number of repetitions, and the lower to denote the quantity for which the substitution is made, thus

$$\psi^{2,\,2,\,1}_{2\;\;1\;\;2}(x,y)=\psi\Big\{\psi\{x,\,\psi(x,y)\},\ \psi\big[\psi\{x,\,\psi(x,y)\},\ \psi(x,y)\big]\Big\},$$

when these indices become numerous it is desirable, for the sake of facility in printing, to have them on a level with the rest of the symbols, and this will become almost indispensible if another or possibly two other rows of indices should be employed: the further extension of the calculus may require such additions to indicate modifications of a different kind from those yet considered. There appear to be four circumstances connected with the indices of functions whose relations it may ultimately be desirable to express by means of them: the two first of them, namely, the number of repetitions and the quantities for which substitutions are made are already pointed out by the two rows. It may become necessary to substitute

instead of one of the variables, not the original function but some modification of it such as $\psi^{1,2,}(x,y)$, and this might be only substituted for one of the variables in certain particular places not universally; such changes would require two additional rows.

The alteration which I would propose in all cases where many indices are concerned is extremely simple: it consists in merely bringing the indices down to the level of the functional sign and inclosing them between two bars; the expression last-mentioned would be written thus

$$\psi\left|\begin{smallmatrix}2, & 2, & 1,\\ 2 & 1 & 2\end{smallmatrix}\right|(x, y).$$

This would also possess some advantage in the case of a function of many variables being after the performance of given operations reduced to a function of a less number, (by means of some assigned relation between some of them) and then new functional operations being performed, or considered as a function of the reduced number of variables, thus

$$\psi\left|\begin{smallmatrix}2, & 1, & 1\\ 1 & 2 & 3\end{smallmatrix}\right|\begin{smallmatrix}2, & 2\\ 1 & 2\end{smallmatrix}\Big|.(x, y, z)\ldots\ldots\ldots\{z=f(x, y)\},$$

which signifies that the second function of $\psi(x, y, z)$ is taken relative to x; $f(x, y)$ is then substituted for z and the second function of the result considered as a function of two variables is taken first with respect to x, then with regard to y.

The method I have pointed out for the determination of the number of times the quantities under the functional sign would be repeated if they were written out at length is perhaps the most direct and unartificial which could be proposed; I cannot however terminate this paper without pointing out another method of a character completely opposite which promises in more complicated enquiries of this nature to be of considerable use. It bears a strong resemblance to a very elegant artifice employed by M. Laplace in order to determine the numerical coefficients of Σu, and was in fact suggested by it.

Since the number of times any variable is repeated is independent of the form* of the function, that number must be the same for all functions. If therefore we can find it for any particular function, we have it for all others. Let us then select the function $x_1 + x_2 + \ldots x_n$ and take those functions which the index requires, we shall have the number of repetitions of each variable expressed by the coefficient attached to it.

One great advantage of this plan is that we always have it in our power to take any function however complicated of $x_1 + x_2 + \ldots x_n$; as an example let it be required to find the number of repetitions of x and y in $\psi^{a, b,}(x, y)$;

$$\text{if } \psi(x, y) = vx + uy,$$

$$\psi^{a, b}(x, y) = \left(v^a + uv\frac{v^a - 1}{v - 1} \cdot \frac{u^{b-1} - 1}{u - 1}\right) x + u^b \cdot \frac{v^a - 1}{v - 1} y;$$

if $u = v = 1$, this becomes

$$\psi^{a, b}(x, y) = (1 + a.\overline{b-1})x + ay = (ab - a + 1)x + ay.$$

Or x will be found repeated $ab - a + 1$ times, and y will occur a times, which coincides with what was proved in Prob. 7.

The surprising condensation of meaning comprised in small space and yet exempt from even the slightest tinge of obscurity is nowhere more conspicuous than it is throughout the calculus of functions: and the solution of the problems contained in this paper enables us to give numerical results which will be viewed with surprise even by those who are best acquainted with the power of general signs.

The equation,

$$\psi^{\overline{10.10},}(x, y) = \psi(x, y),$$

is one whose solution presents no great difficulties and may be

* In whatever manner the original function may involve the unknown quantities it must in this point of view be considered as containing each only once.

accomplished in a few lines. To any person acquainted with the notation employed in the doctrine of functions the question is comprehended at a single glance; yet if we apply to it the rules discovered in Prob. 1 and 2, we shall find that if it were written out at length, x and y would each be repeated 512 times, and ψ would occur 1023 times; so that the whole expression would consist of 2047 letters, and it may be added, that if it were so developed it would require a much longer time merely to comprehend the enunciation of the problem than it would to understand and solve it in its contracted form.

CHARLES BABBAGE.

Feb 26, 1820.